Technical Notes

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A Generalized Quadrature Formula for Cauchy Integrals

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CAUCHY integrals, which are common in aerodynamics, cannot always be evaluated in closed form. There is a demand, therefore, for numerical methods. Such methods have been proposed by e.g., Longman,[1] Song,[2] Garrick,[3] and Brakhage.[4] The first two are similar to the method of subtraction of the singularity,[5] while Garrick's method is based on the calculation of the conjugate Fourier series, which is done in an optimal way. Using Garrick's result, Brakhage gave a corresponding explicit quadrature formula for a Cauchy integral (i.e., a Cauchy principal value) of the form

$$\oint_0^{\pi} \frac{f(\psi)}{\cos\phi - \cos\psi} d\psi \tag{1}$$

This was based on the condition that $f(\phi)$ is an even, 2π-periodic function of ϕ. For $f(\phi)$ analytic, he also estimated the error.

The applicability of Brakhage's formula can be extended. We show here that a corresponding formula can be derived for integrands containing an arbitrary weight function. We consider the integral

$$\oint_{-1}^{1} u(y)(y - x)^{-1}dy \tag{2}$$

and assume that $u(x)$ is a product of a regular factor and the weight function $W(x)$; this is positive and integrable but not necessarily regular. We prove that the formula

$$\oint_{-1}^{1} u(y)(y - X_m)^{-1}dy = \sum_{n=1}^{N} a_n u(x_n)(X_n - X_m)^{-1} \tag{3}$$

is exactly valid under the following conditions:

1) The ratio $u(x)/W(x)$ is a polynomial of degree $\leqq 2N$.

2) The points $x = x_n$, $n = 1(1)N$, are the N zeros of the polynomial $P_N(x)$ of degree N in the system $\{P_n(x)\}$ of orthogonal polynomials assuming $W(x)$ as weight function, $-1 < x < 1$.

3) The points $x = X_m$, $m = 1, 2, \ldots$, are zeros of the function

$$Q_N(x) = -\tfrac{1}{2} \oint_{-1}^{1} W(y)P_N(y)(y - x)^{-1}dy \tag{4}$$

4) The coefficients a_n are defined by

$$a_n = -2Q_N(x_n)/W(x_n)P_N'(x_n) \tag{5}$$

where $P_N'(x_n)$ is the derivative of $P_N(x)$ at $x = x_n$.

The weights a_n and the abscissas x_n are seen to be identical with those of the ordinary Gaussian formula for the weight function $W(x)$. The zeros $x = X_m$ of $Q_N(x)$ are regarded as optimal locations, for, with $2N + 1$ parameters (a_n, x_n, and X_m) at our disposal, it should be possible to set up a quadrature formula which yields exact results for arbitrary polynomials of degree $2N$, but not for those of higher orders.

In order to prove Eq. (3), we apply it to a test function $u_1(x)$. By means of the interpolation functions

$$h_n(x) = P_N(x)W(x)/P_N'(x_n)W(x_n)(x - x_n) \tag{6}$$

and an arbitrary polynomial of degree N,

$$A + (x - X_m)F_{N-1}(x)$$

we define $u_1(x)$ as

$$u_1(x) = \sum_{n=1}^{N} u(x_n)h_n(x) + W(x)P_N(x)[A + (x - X_m)F_{N-1}(x)] \tag{7}$$

Equation (3) is valid, if it is valid for $u_1(x)$, for $u_1(x)/W(x)$ is an arbitrary polynomial of degree $2N$.

Using the relation

$$a_n = (x_n - X_m) \oint_{-1}^{1} h_n(y)(y - X_m)^{-1}dy \tag{8}$$

derived below and inserting $u_1(y)$ for $u(y)$, we recast the left-hand member of Eq. (3) in the form

$$\sum_{n=1}^{N} a_n u(x_n)(x_n - X_m)^{-1} + A \oint_{-1}^{1} \times W(y)P_N(y)(y - X_m)^{-1}dy + \oint_{-1}^{1} W(y)P_N(y)F_{N-1}(y)dy \tag{9}$$

The first integral in this expression is zero, for the points $x = X_m$ were defined as zeros of $Q_N(x)$. The second integral is also zero, for $F_{N-1}(x)$ may be regarded as a linear combination of the polynomials $P_n(x)$, $n = 0(1)(N - 1)$, which are orthogonal to $P_N(x)$. This nearly completes the proof, for the remaining summation agrees with the right-hand member of Eq. (3).

The relation (8) may be proved by inserting Eq. (6) for $h_n(y)$. Decomposing the integrand into partial fractions and noting that owing to the definition of X_m one of the integrals appearing is zero, we arrive at the expression

$$a_n = \frac{1}{W(x_n)P_N'(x_n)} \oint_{-1}^{1} W(y)P_N(y)(y - x_n)^{-1}dy = \oint_{-1}^{1} h_n(y)dy \tag{10}$$

This agrees with Eq. (5).

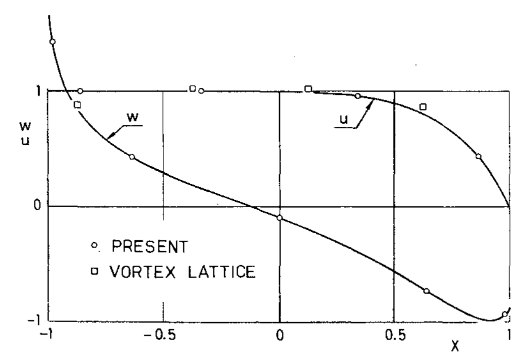

Fig. 1 $u = 1 - [(1 + x)/2]^8$ **and corresponding** w **function.**

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Application

We consider two-dimensional flow about a thin profile and let the values at the surface of the tangential and normal velocities of a lifting flow (or the normal and tangential velocities of a nonlifting flow) be denoted by $u(x)$ and $w(x)$, respectively. These quantities are approximately related by

$$\oint_{-1}^{1} u(y)(y - x)^{-1}dy = \pi w(x) \tag{11}$$

Applying the quadrature formula (3) to the Cauchy integral in Eq. (11), we immediately obtain the discretized counterpart

$$\sum_{n=1}^{N} a_n u(x_n)(x_n - X_m)^{-1} = \pi w(X_m) \qquad m = 1,2, \ldots \tag{12}$$

If $u(x)/W(x)$ is a polynomial of degree $\leqq 2N$, the values $u(x_n)$ and $w(X_m)$ are exact values of functions $u(x)$ and $w(x)$ related by Eq. (11).

We illustrate this in Fig. 1 in the case of $W(x) = 1$, i.e., in a case where the tangential velocity itself (or the normal velocity of a nonlifting flow) is a polynomial. The two solid curves represent the function $u(x) = 1 - [(1 + x)/2]^8$ and the corresponding w-function, which is defined by Eq. (11). We have chosen $N = 4$. Therefore, and as the order of the polynomial considered is equal to the maximum order $2N = 8$, the relation (12) is exactly valid. The points $[X_m,w(X_m)]$ and $[x_n,u(x_n)]$ must accordingly lie on the solid curves, as is shown by the circles in the figure.

It is interesting to compare the present discretized method with the vortex-lattice method. The latter employs as integration points (vortex points) and singularity locations (control points) the $\frac{1}{4}$ and $\frac{3}{4}$ points on N equal subintervals of the chord. It is generally known that these locations yield surprisingly good results in ordinary cases. In the present example, by using w values agreeing with the w curve, they yield u values which have been marked by squares in the figure.

A similar example but for constant loading has been treated by James[6] by the vortex-lattice method. His results deviate, however, rather much (about 9% for $N = 20$) from the true solution. As the u function considered by James is nonzero at the trailing edge, while the present one vanishes there, we are tempted to conclude that the vortex-lattice theory is unsuitable for functions which do not satisfy the Kutta-Joukowsky condition.

In the ordinary case of a lifting flow with constant or polynomial downwash, it is appropriate to choose $W(x) = [(1 - x)/(1 + x)^{1/2}]$, and for a slender or a finite wing it is also relevant to consider the weight functions $(1 - x^2)^{-1/2}$ and $(1 - x^2)^{1/2}$. Examples for these are not given here, for the formula (3) is then equivalent with the formula of Brakhage, and applications of this were shown in Ref. 7. Borja and Brakhage[7] achieved such a transformation of the basic integral relation that the quadrature formula can be applied also for a finite wing for chord-wise and span-wise integration. In a similar way, Borja[8] treated recently even the problem of an oscillating wing in incompressible flow.

We finally note that, if it is required to evaluate a weighted integral

$$L = \int_{-1}^{1} H(x)u(x)dx \tag{13}$$

as is often the case in aerodynamics, this can be done in a straightforward way by means of the Gaussian formula

$$L \approx \sum_{n=1}^{N} H(x_n)a_n u(x_n) \tag{14}$$

This does not require knowledge of a_n as the products $a_n u(x_n)$ can be solved from Eq. (12). The formula (14) yields exact values for L if $H(x)$ is a polynomial and if the degree of this plus the degree of $u(x)/W(x)$ is less than $2N$. Hence, the expressions (12) and (14) may be said to represent a generalization of the $\frac{3}{4}$ chord point formula of Pistolesi,[9] but not only to any number of vortices and control points but also to loadings characterized by an arbitrary weight function $W(x)$.

Conclusions

It has been demonstrated that the Gaussian quadrature formulas can be applied in the ordinary way even to a Cauchy integral if the singularity is located at any of certain appropriate points. Such points have been defined for integrands containing an arbitrary weight function and a regular factor. The formulas are exactly valid for polynomial factors of a degree equal to twice the number of integration points.

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Wave Propagation in Three-Layered Plates

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CONSIDER a traction-free, infinitely long plate made of three layers of different isotropic materials of different thickness (see Fig. 1). The material constants and geometry pertinent to layers 1, 2, or 3 will be designated by the superscripts (1), (2), or (3), respectively.

The solution of the Navier equations of motion for the ith

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